Linear Algebra I

30/01/2018, Tuesday, 14:00 – 17:00

You are **NOT** allowed to use any type of calculators.

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1 Linear systems of equations $(2+5+3+2+3=15$ pts$)$

Consider the linear system of equations

$$\begin{array}{rcrcrcl} 2x & - & y & + & 3z & = & a \\ x & - & y & + & z & = & b \\ 7x & - & 2y & + & 12z & = & c \end{array}$$

in the unknowns x, y, and z.

(a) Write down the corresponding augmented matrix.

(b) Put it into the row echelon form.

(c) Determine the values of a, b, and c such that the system is consistent.

(d) Determine the lead and free variables when the system is consistent.

(e) Find the solution set.

2 Determinants $(5+6+4=15$ pts$)$

Consider the $n \times n$ matrix of the form

$$B_n = \begin{bmatrix} 2 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 2 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 2 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 2 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 2 \end{bmatrix}.$$

Let $D_n = \det(B_n)$.

(a) Find D_1, D_2, and D_3.

(b) Expand along the first row and find numbers a and b such that $D_n = aD_{n-1} + bD_{n-2}$ for $n \geqslant 3$.

(c) Prove by induction on n that $D_n = n + 1$ for $n \geqslant 3$.

3 Vector spaces (7 + 8 = 15 pts)

(a) Let $X \in \mathbb{R}^{n\times n}$ and $S(X) = \{A \in \mathbb{R}^{n\times n} \mid AX + XA = 0\}$. Show that $S(X)$ is a subspace of $\mathbb{R}^{n\times n}$. For $n = 3$ and $X = \left[\begin{smallmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right]$, find a basis for $S(X)$ and find its dimension.

(b) Let a and $b \neq 0$ be real numbers. Consider the functions $f(x) = e^{ax}\sin(bx)$, $g(x) = e^{ax}\cos(bx)$ and the subspace $S = \operatorname{span}(f, g)$ of the vector space of continuous functions. Let $L : S \to S$ be given by $L(h) = h''$ where h'' denotes the second derivative of h. Show that L is a linear transformation. Find its matrix representation relative to the basis (f, g).

4 Least squares problem (15 pts)

Find the parabola of the form $y = a + bx^2$ that gives the best least squares approximation to the points:

x	0	1	2
y	0	2	3

5 Characteristic polynomial, determinant, and trace (5 + (1 + 2 + 3 + 4) = 15 pts)

(a) Let $M \in \mathbb{R}^{n\times n}$ and $p(s) = p_k s^k + p_{k-1}s^{k-1} + \cdots + p_1 s + p_0$ be a polynomial. Show that if λ is an eigenvalue of M then $p(\lambda)$ is an eigenvalue of $p(M) = p_k M^k + p_{k-1}M^{k-1} + \cdots + p_1 M + p_0 I_n$.

(b) Let $A \in \mathbb{R}^{n\times n}$ be such that $A^2 - A + I_n = 0$.

(i) Show that A is nonsingular.

(ii) Show that n is an even number.

(iii) Show that the characteristic polynomial of A is $(\lambda^2 - \lambda + 1)^{\frac{n}{2}}$.

(iv) Show that $\operatorname{tr}(A) = \frac{n}{2}$ and $\det(A) = 1$.

6 Eigenvalues/vectors (1 + 2 + 4 + 4 + 4 = 15 pts)

Consider the matrix

$$M = \begin{bmatrix} a & 2b & 0 \\ b & a & b \\ 0 & 2b & a \end{bmatrix}$$

where a and $b \neq 0$ are real numbers.

(a) Show that a is an eigenvalue of M.

(b) Find an eigenvector corresponding to the eigenvalue a.

(c) Find other eigenvalues of M and corresponding eigenvectors.

(d) Determine the values of a and b such that M is nonsingular.

(e) Is M diagonalizable? If so, find a diagonalizer.

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10 pts free